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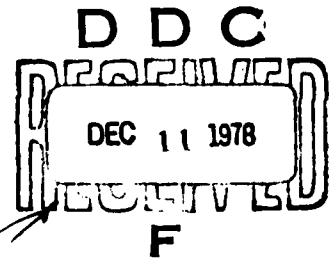
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FREQUENCY RESPONSE OF  
 $\alpha$ - $\beta$ - $\gamma$  TRACKERS

SEPTEMBER 1978

FINAL REPORT



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**Block 20. ABSTRACT (Cont'd)**

( $\omega$  is the radian frequency and  $T$  is the sample interval). After this change, the magnitude and phase for various output-input ratios is plotted as a function of  $\omega T$ . Linear plots and Bode plots (especially useful for real-time control system design) are presented. Equations are given for the frequency at which the output is 0.707 of the input, for the minimum of the output to input ratio and for the overshoot (amount by which the output exceeds the input).

A synopsis of previously available statistical properties of  $\alpha$ - $\beta$ - $\gamma$  trackers is also included.

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## INTRODUCTION

The  $\alpha$ - $\beta$ - $\gamma$  tracker is a system of equations which use estimates of position, velocity, and acceleration to predict a new (updated) position, a new velocity and a new acceleration. Then the error between the measured position and the predicted position is used to improve (correct) the estimates. The parameter  $\gamma$  is associated with acceleration and if it is set to zero the  $\alpha$ - $\beta$ - $\gamma$  tracker becomes an  $\alpha$ - $\beta$  tracker. In  $\alpha$ - $\beta$  trackers, acceleration is not estimated nor used in the predictor equations.

The  $\alpha$ - $\beta$ - $\gamma$  tracker is especially well suited for computer algorithms. The  $\alpha$ - $\beta$ - $\gamma$  algorithms require very little storage and a small number of computations and are well suited for real-time applications.

The Kalman tracking algorithms require much more storage and many more computations. However, the Kalman algorithms produce estimates with minimum rms error estimates given non-stationary measurement noise.

The standard  $\alpha$ - $\beta$ - $\gamma$  tracker minimizes the exponentially weighted squared error between the data and parabolic line through the data.

The QD based  $\alpha$ - $\beta$ - $\gamma$  tracker minimizes the squared error between the data and a parabolic line through the data--over some number of points and subject to slope and intercept conditions.

## THE $\alpha$ - $\beta$ - $\gamma$ TRACKING EQUATIONS

The  $\alpha$ - $\beta$ - $\gamma$  tracking Equations (1) through (6) are:

Corrector (or Estimator) Equations:

$$\bar{x}_n = x_{np} + \alpha(x_n - x_{np}) \quad (1)$$

$$\bar{v}_n = v_{np} + \frac{\beta}{T} (x_n - x_{np}) \quad (2)$$

$$\bar{a}_n = a_{np} + \frac{\gamma}{T^2} (x_n - x_{np}) \quad (3)$$

Predictor Equations:

$$x_{np} = \bar{x}_{n-1} + \bar{v}_{n-1}T + \frac{1}{2} \bar{a}_{n-1}T^2 \quad (4)$$

$$v_{np} = \bar{v}_{n-1} + \bar{a}_{n-1}T \quad (5)$$

$$a_{np} = \bar{a}_{n-1} \quad (6)$$

where

$x_n$  = measured quantity,

$\bar{x}_n, \bar{v}_n, \bar{a}_n$  = estimated quantities,

$x_{np}, v_{np}, a_{np}$  = predicted quantities,

$\bar{x}_{n-1}, \bar{v}_{n-1}, \bar{a}_{n-1}$  = estimates from previous step, and

$T$  = step size.

The Equations (1) through (6) are in a form that can be efficiently programmed on a computer.

Substituting Equations (5) and (6) into (2) and (3) yields another form:

$$\bar{x}_n = x_{np} + \alpha(x_n - x_{np}) \quad (7)$$

$$\bar{v}_n = \bar{v}_{n-1} + \bar{a}_{n-1}T + \frac{\beta}{T} (x_n - x_{np}) \quad (8)$$

$$\bar{a}_n = \bar{a}_{n-1} + \frac{\gamma}{T^2} (x_n - x_{np}) \quad (9)$$

$$x_{np} = \bar{x}_{n-1} + T\bar{v}_{n-1} + \frac{1}{2} \bar{a}_{n-1}T^2 \quad (10)$$

#### PERFORMANCE MEASURES [1]

One performance measure is the ratio of output variance to input variance with noise as the only input. These ratios are defined by

- [1] H. R. Simpson, "Performance Measures and Optimization Condition for a Third Order Sampled Data Tracker," Transactions on Automatic Control, April 1963, pp. 182-183.

$$k_x(0) = \frac{\text{steady-state variance in position output}}{\text{variance in raw position input}} ,$$

$$k_v(0) = \frac{\text{steady-state variance in velocity output}}{\text{variance in raw position input}} ,$$

$$k_a(0) = \frac{\text{steady-state variance in acceleration output}}{\text{variance in raw position input}} .$$

The equations for these variance reduction factors are

$$k_x(0)Y^2 = 2\beta\gamma(2\alpha^2 + 2\beta - 3\alpha\beta) - \alpha\gamma^2(4 - 2\alpha - \beta) , \quad (11)$$

$$k_v(0)Y^2T^2 = 4\beta^3\gamma - 4\beta^2\gamma^2 + 2\gamma^3(2 - \alpha) , \quad (12)$$

$$k_a(0)Y^2T^4 = 4\beta\gamma^3 , \quad (13)$$

where

$$Y^2 = \gamma[\alpha(2\beta + \gamma) - 2\gamma](4 - 2\alpha - \beta) . \quad (14)$$

One transient performance measure  $D^2$  is obtained from the response of the tracker to a unit step input of velocity. Ideally, the position output would be a ramp, the velocity output would be a constant, and the acceleration output would be zero for  $t > 0$ . The  $D^2$  measures are the sums of the squared error between the ideal response and the actual response:

$$D_x^2Y^2 = 2\gamma(2 - \alpha)(1 - \alpha)^2 , \quad (15)$$

$$D_v^2Y^2T^2 = 2\gamma[\alpha^2(2 - \alpha) + 2\beta(1 - \alpha)] + (\gamma^3/8)(2 - \alpha) - (\gamma^2/4)[2(4 - 3\alpha)(2 - \alpha) + \alpha\beta] , \quad (16)$$

$$D_a^2Y^2T^4 = \alpha\gamma^2(4 - 2\alpha - \beta) + (\gamma^3/2)(2 - \alpha) . \quad (17)$$

The final performance measure  $A^2$  is obtained from the response of the tracker to a unit step input of acceleration. Ideally, the position output would

be a parabola, the velocity output would be a ramp and the acceleration output would be a constant. The  $A^2$  measures are sums of the squared error between the ideal response and the actual response:

$$A_x^2 y^2 = \alpha(4 - 2\alpha - \beta)(1 - \alpha)^2, \quad (18)$$

$$A_v^2 y^2 T^2 = (\alpha/4)(\beta - 2\alpha)^2(4 - 2\alpha - \beta) + (\gamma/8)(2 - \alpha)(4 - 2\alpha - \beta)^2, \quad (19)$$

$$A_x^2 y^2 T^4 = \alpha\beta^2(4 - 2\alpha - \beta) + (\gamma/2)[2\alpha^2 - (2 - \alpha)\beta](4 - 2\alpha - \beta). \quad (20)$$

Finally, Simpson determined that all these performance measures are minimized if

$$2\beta - \alpha(\alpha + \beta + \gamma/2) = 0. \quad (21)$$

Note that Equation (21) is one equation in three unknowns and does not completely specify the design.

#### "STANDARD" $\alpha$ - $\beta$ - $\gamma$ TRACKER

If the error between the sampled data and a parabolic line approximating the data is squared, weighted exponentially, and minimized,  $\alpha$ ,  $\beta$ , and  $\gamma$  are found [2] in terms of a parameter,  $\theta$ . Kahrlas calls this tracker "critically damped."

$$\alpha = 1 - \theta^3 \quad (22)$$

$$\beta = \frac{3}{2} (1 - \theta)^2 (1 + \theta) \quad (23)$$

$$\gamma = (1 - \theta)^3 \quad (24)$$

Morrison [3] calls such a tracker "a fading memory polynomial filter of degree 2." Morrison [4] gives the variance reduction factors in terms of  $\theta$ .

[2] P. J. Kahrlas, Electronic Scanning Radar Systems Design Handbook, Artech House, 1976, p. 215.

[3] Norman Morrison, Introduction to Sequential Smoothing and Prediction, McGraw-Hill, 1969, p. 516.

[4] Morrison, p. 526.

$$k_x(0) = \frac{(1 - \theta)}{(1 + \theta)^5} (19 + 24\theta + 16\theta^2 + 6\theta^3 + \theta^4) \quad (25)$$

$$k_v(0)T^2 = \frac{(1 - \theta)^3}{(1 + \theta)^5} \left( \frac{49 + 50\theta + 13\theta^2}{2} \right) \quad (26)$$

$$k_a(0)T^4 = 6 \frac{(1 - \theta)^5}{(1 + \theta)^5} \quad (27)$$

#### QD TRACKER [5]

The QD tracker minimizes the sum of the squared error between  $m$  data points and a polynomial passing through the data--subject to intercept and slope constraints. For a "second order" QD, the polynomial is a quadratic and [6]

$$\alpha = \frac{60m^2}{10m^3 + 33m^2 + 23m - 6}, \quad (28)$$

$$\beta = \frac{2\alpha}{m}, \quad (29)$$

$$\gamma = \frac{2\alpha}{m^2}. \quad (30)$$

#### $\alpha$ - $\beta$ TRACKER

The system of Equations (1) through (6) can be converted to an  $\alpha$ - $\beta$  tracker by setting  $\gamma = 0$ . Then all of Simpson's results reduce to those given by Benedict and Bordner [7].

If the error between the sampled data and a straight line approximating the data is squared, weighted exponentially, and minimized,  $\alpha$  and  $\beta$  are found (see Reference [2]) in terms of  $\theta$ .

- [5] W. A. McCool, "A Matrix-Vector Formulation of the QD Digital Filter," Technical Report, WSMR, New Mexico, December 1968.
- [6] W. L. Shepherd, Draft Report, Instrumentation Directorate, WSMR, New Mexico.
- [7] T. R. Benedict and G. W. Bordner, "Synthesis of an Optimal Set of Radar Track While Scan Smoothing Equations," Transactions on Automatic Control, July 1962, pp. 27-32.

$$\alpha = 1 - \theta^2$$

$$\beta = (1 - \theta)^2$$

Morrison (see Reference [3]) calls such a tracker "a fading memory polynomial of degree 1." He gives (see Reference [4]) the variance reduction factors in terms of  $\theta$ .

$$k_x(0) = \frac{1 - \theta}{(1 + \theta)^3} (5 + 4\theta + \theta^2) \quad (31)$$

$$k_y(0)T^2 = \frac{2(1 - \theta)^3}{(1 + \theta)^3} \quad (32)$$

#### Z TRANSFORMED $\alpha$ - $\beta$ - $\gamma$ TRACKER

Substituting Equation (10) into Equations (7) through (9) and taking the Z transform yields

$$(Z + \alpha - 1)\bar{X}_n + (\alpha - 1)T\bar{V}_n + [(\alpha - 1)/2]T^2\bar{A}_n = \alpha ZX_n, \quad (33)$$

$$\beta\bar{X}_n + (Z + \beta - 1)T\bar{V}_n + (\beta/2 - 1)T^2\bar{A}_n = \beta ZX_n, \quad (34)$$

$$\gamma\bar{X}_n + \gamma T\bar{V}_n + (Z + \gamma/2 - 1)T^2\bar{A}_n = \gamma ZX_n. \quad (35)$$

Writing Equations (33) through (35) in matrix form and using Cramer's rule yields

$$\frac{\bar{X}_n}{X_n} = \frac{Z[\alpha(Z - 1)^2 + (\beta + \gamma/2)(Z - 1) + \gamma]}{\Delta}, \quad (36)$$

$$\frac{\bar{V}_n}{X_n} = \frac{Z[\beta(Z - 1)^2 + \gamma(Z - 1)]}{T\Delta}, \quad (37)$$

$$\frac{\bar{A}_n}{X_n} = \frac{Z\gamma(Z - 1)^2}{T^2\Delta}, \quad (38)$$

$$\frac{x_{np}}{x_n} = \frac{(\alpha + \beta + \gamma/2)(Z - 1)^2 + (\beta + 3/2 \gamma)(Z - 1) + \gamma}{\Delta} , \quad (39)$$

where

$$\Delta = (Z - 1)^3 + \alpha(Z - 1)^2 + (\beta + \gamma/2)Z(Z - 1) + \gamma Z . \quad (40)$$

### STABILITY

For an  $\alpha$ - $\beta$  tracker, the characteristic equation is [8]

$$\Delta = (1 - \alpha) + (\alpha + \beta - 2)Z + Z^2 . \quad (41)$$

Stability requirements are [8]

$$\alpha > 0 , \quad (42)$$

$$\beta > 0 , \quad (43)$$

$$2\alpha + \beta < 4 . \quad (44)$$

Stability constraints for an  $\alpha$ - $\beta$ - $\gamma$  tracker may be obtained by expanding Equation (40) and applying the constraints given in Jury [9]:

$$0 < \alpha < 2 , \quad (45)$$

$$2\alpha + \beta < 4 , \quad (46)$$

$$\alpha\beta + \alpha\gamma/2 - \gamma > 0 , \quad (47)$$

$$\gamma > 0 . \quad (48)$$

Note that Inequalities (42), (43), and (44) imply Expression (45), and that Expressions (45), (47), and (48) imply that  $\beta > 0$  (Expression (43)). Finally, observe that setting  $\gamma = 0$  reduces the  $\alpha$ - $\beta$ - $\gamma$  expressions to the  $\alpha$ - $\beta$  results. For the  $\alpha$ - $\beta$ - $\gamma$  tracker, Relations (45) through (48) are necessary and sufficient. Expression (47) may be replaced by the more convenient sufficient (but not necessary) condition of (49):

$$\alpha\beta > \gamma . \quad (49)$$

[8] R. L. Gonzales, "Performance Models for Range and Angle Trackers Applicable to Phased Array Instrumentation Radar," Electronics Division Technical Memorandum 68-2, Instrumentation Directorate, WSMR, New Mexico, June 1968.

[9] E. I. Jury, Theory and Application of the Z-Transform, John Wiley, 1964, p. 93.

## FREQUENCY RESPONSE CURVES

To find the frequency response for any one of the transfer functions (36) through (40), replace  $Z$  by  $e^{+j\omega T}$  and plot the magnitude and phase. (Recall that the frequency response of digital filters is periodic with period  $2\pi$  in  $\omega T$ . Only a half period is shown in the plot.)

A linear plot of  $|\bar{X}_n/X_n|$  and phase of  $\bar{X}_n/X_n$  is shown in Figure 1. The same plot on a log-log scale is shown in Figure 2. Note that the curves of Figure 2 are Bode plots and can be used in control system design. A design example for using Figure 2 follows.

### DESIGN EXAMPLE

1. Select the sample time for a standard (critically damped)  $\alpha$ - $\beta$ - $\gamma$  tracker with  $\alpha = .1$  if the tracker must pass 5 Hz.
2. Determine the attenuation of a 60 Hz signal for this sample time.
3. Determine the frequency of the first maximum response above the desired pass band.

### SOLUTION

1. Use 3 dB down as the edge of the pass band. From the graph,  $x(.1)$  is 3 dB down at  $\omega T = .135$ .

$$T = \frac{.135}{2\pi(5)} = .0043 \text{ second}$$

Thus, the signal must be sampled 235 times per second (or faster) if 5 Hz is to be attenuated no more than 3 dB.

$$2. \quad \omega T = \frac{60}{5} \times .135 = 1.62$$

From the graph,  $x(.1)$  at  $\omega T = 1.62$  is about 23 dB down.

3. The first undesired maximum occurs at

$$\omega = \frac{2\pi - .04}{T} = 1452/\text{second}$$

or

$$f = 231 \text{ Hz}$$

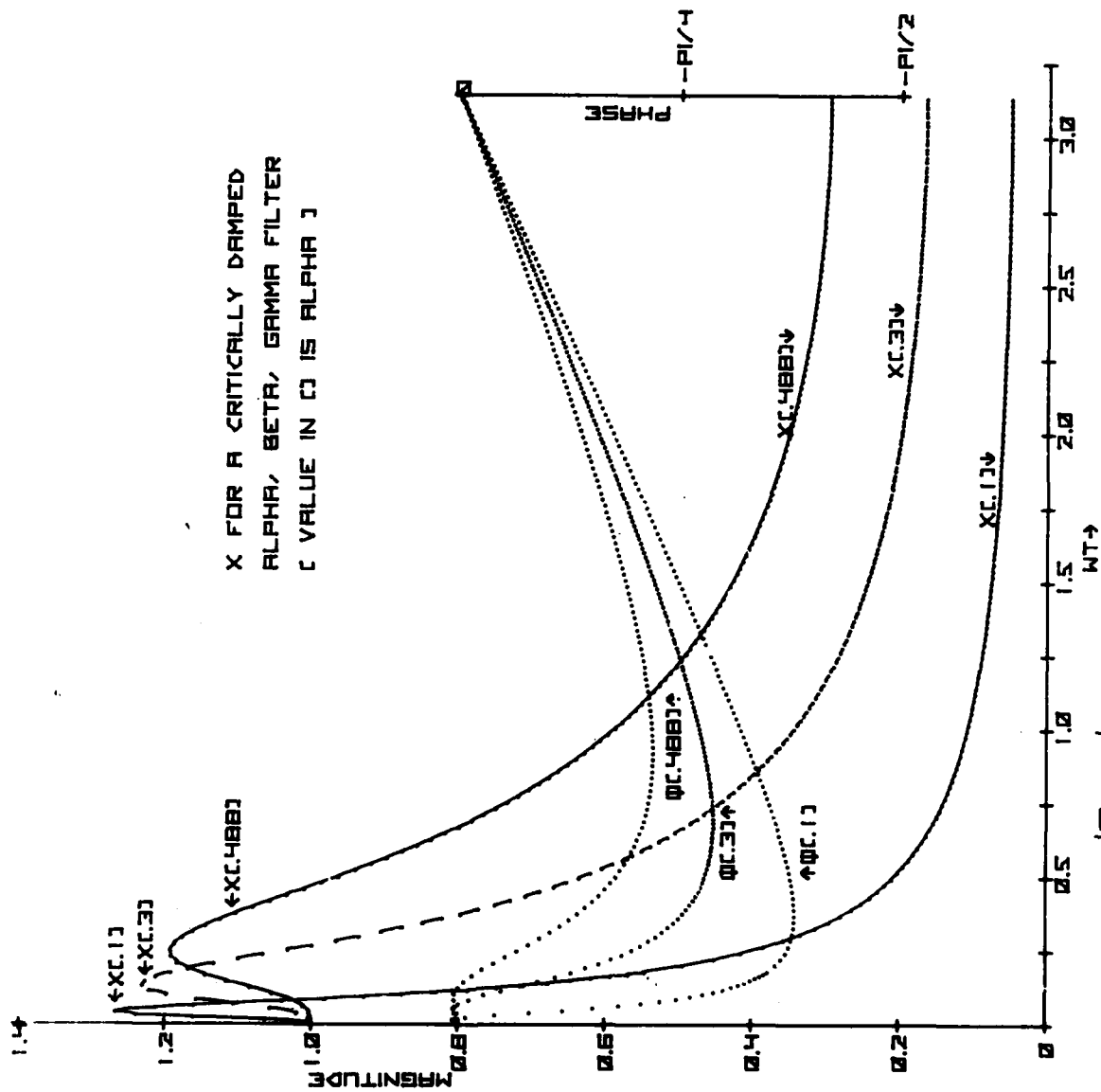


Fig. 1.  $|\bar{X}_n/X_n|$  and phase  $\bar{X}_n/X_n$  for  $Q = .1, .3, .488$

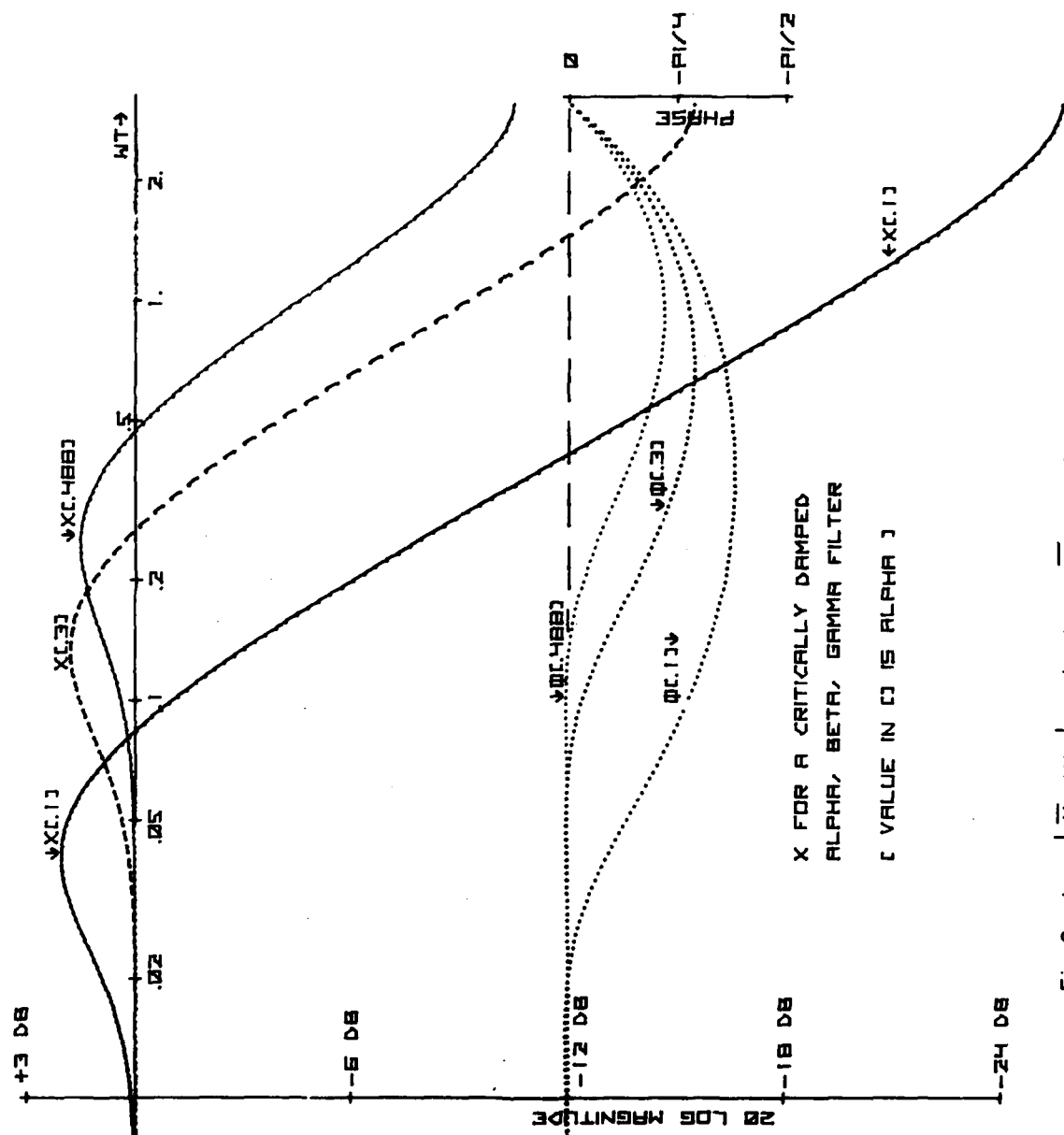


Fig. 2.  $\text{Log } |\bar{X}_n/X_n|$  and phase  $\bar{X}_n/X_n$  for  $Q = .1, .3$  &  $.488$

If the 23 dB of attenuation at 60 Hz is insufficient and a critically damped tracker is desired, then  $\alpha$  must be decreased. Of course, decreasing  $\alpha$  will increase the overshoot.

If the phase characteristic of  $\bar{X}_n/X_n$  is unsatisfactory, then the phase characteristic of  $X_{np}/X_n$  should be examined. Figure 3 reveals that the magnitude curve for  $X_{np}/X_n$  is very similar to the magnitude curve for  $\bar{X}_n/X_n$ , but the phase is monotonically decreasing.

Figure 4 exhibits the magnitude and phase of  $\bar{X}_n/X_n$  and the magnitude and phase of  $200T^2 \bar{A}_n/X_n$  on the same log-log axes. Note that the magnitude characteristics are very similar for  $\omega T$  above 0.2.

Figure 5 compares the QD,  $\alpha$ - $\beta$ , and  $\alpha$ - $\beta$ - $\gamma$  trackers on the same log-log axes. The values of  $\alpha$ ,  $\beta$ , and  $\gamma$  used in these cases are shown in Table 1.

TABLE 1.  $\alpha$ - $\beta$ - $\gamma$  VALUES FOR FIGURE 5

	<u>STANDARD <math>\alpha</math>-<math>\beta</math>-<math>\gamma</math></u>	<u>STANDARD <math>\alpha</math>-<math>\beta</math></u>	<u>QD</u>
$\alpha$	0.3	0.3	0.3
$\beta$	0.0356	0.0267	0.0362
$\gamma$	0.0014	0	0.00219

The  $m$  required to produce  $\alpha = .3$  is 16.6. Table 2 shows the more typical  $m = 21$  values and Figure 6 compares the 3 trackers for the same  $\alpha$ .

TABLE 2.  $\alpha$ - $\beta$ - $\gamma$  VALUES FOR FIGURE 6

	<u>STANDARD <math>\alpha</math>-<math>\beta</math>-<math>\gamma</math></u>	<u>STANDARD <math>\alpha</math>-<math>\beta</math></u>	<u>QD</u>
$\alpha$	0.246	0.246	0.246
$\beta$	0.0231	0.0173	0.0234
$\gamma$	0.00072	0	0.0011

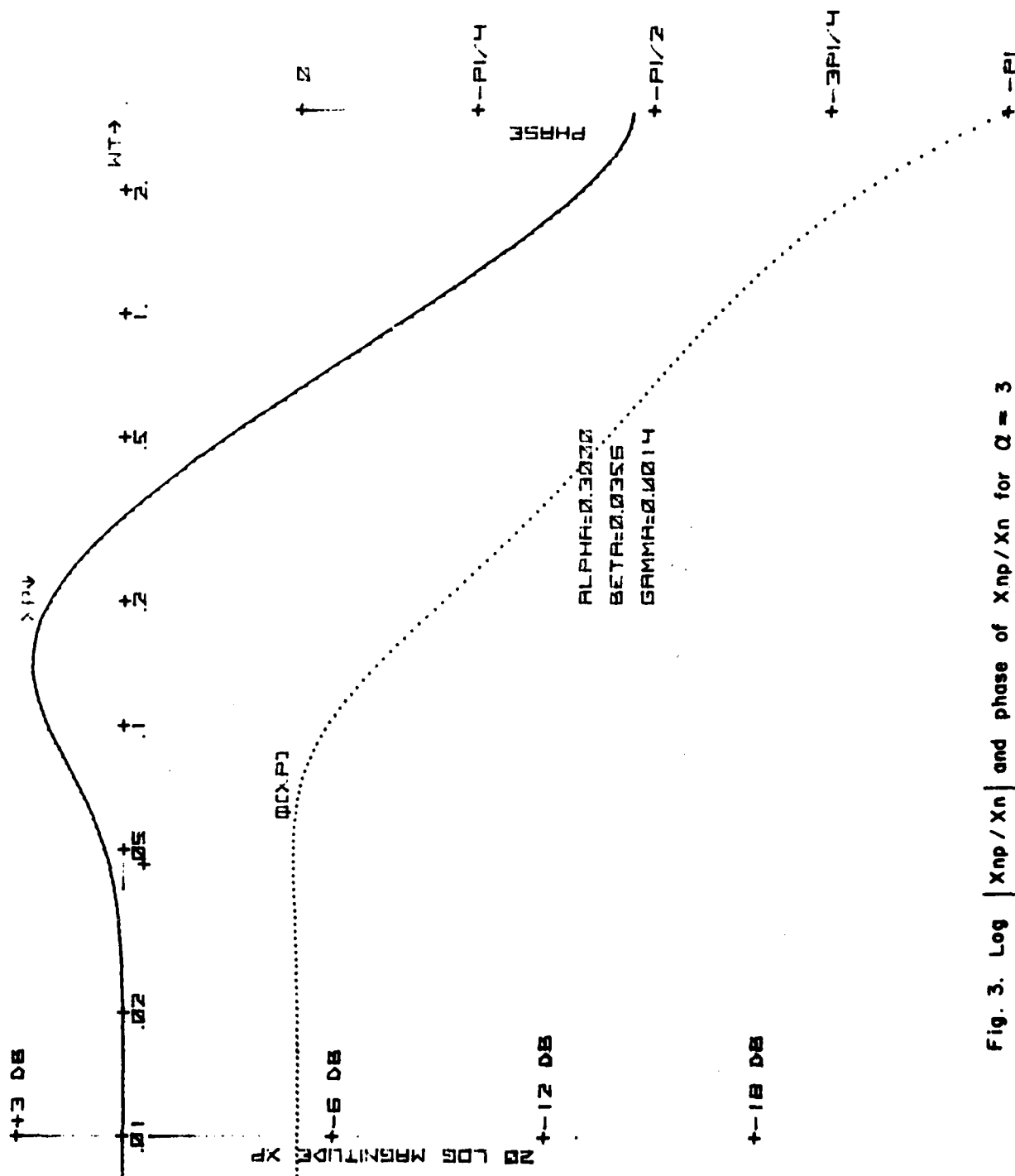


Fig. 3. Log  $|X_{np}/X_n|$  and phase of  $X_{np}/X_n$  for  $Q = 3$





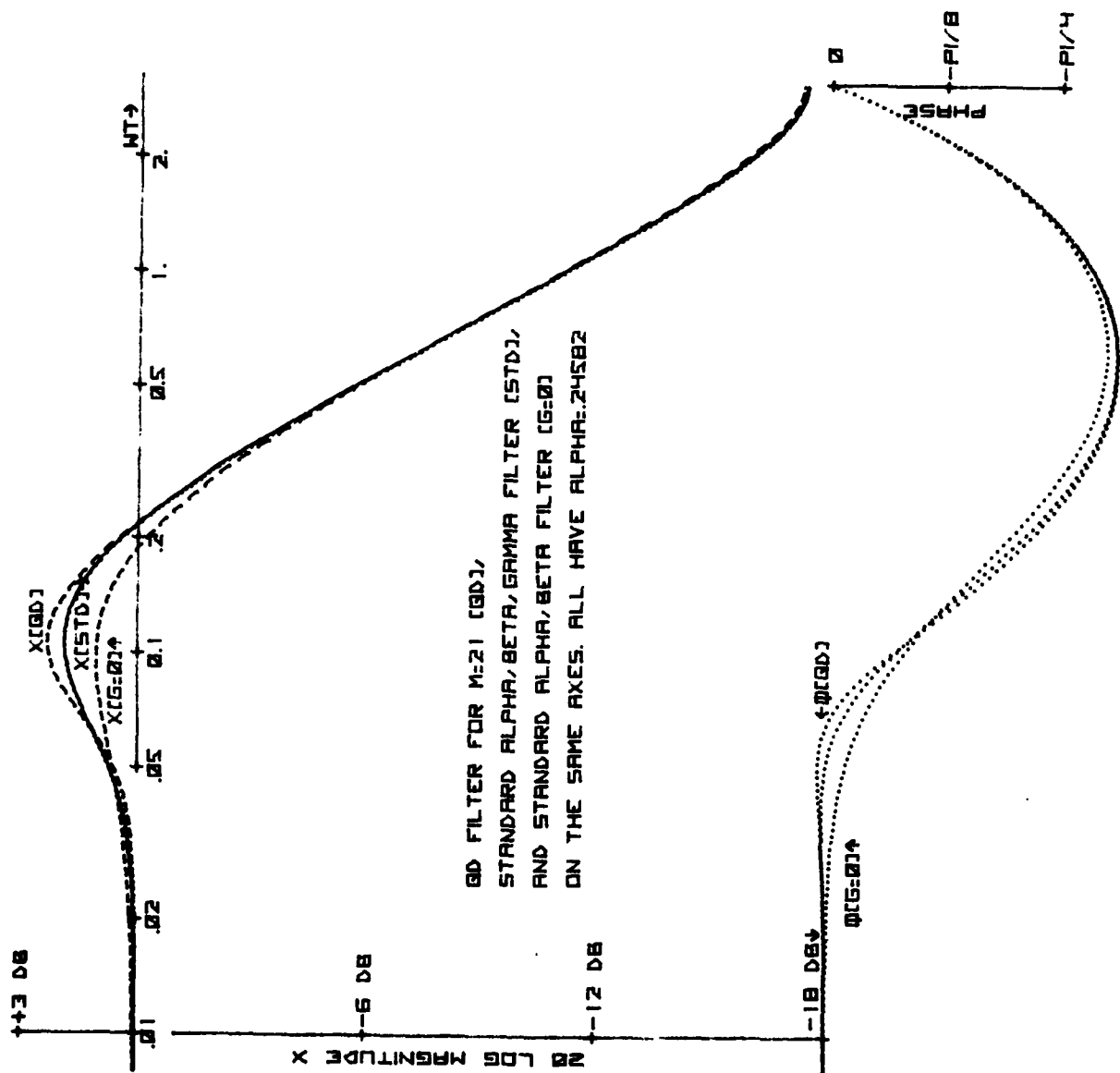


Fig. 6. Comparison of  $Q-\beta-\gamma$ ,  $Q-\beta$  and QD Trackers for  $Q = .246$

### ANALYTICAL EXPRESSIONS

The plots of Figures 1 through 6 contain all the frequency domain information. However, analytic expressions for the minimum and maximum of  $|\bar{X}_n/X_n|$  and the associated frequencies, and the frequency at which  $|\bar{X}_n/X_n| = 0.707$  would be convenient and useful. The minimum of  $|\bar{X}_n/X_n|$  occurs at  $\omega T = \pi$  and is given by Equation (50):

$$\left| \frac{\bar{X}_n}{X_n} \right|_{\min} = \frac{2\alpha - \beta}{4 - 2\alpha - \beta} \quad (50)$$

Note that Equation (50) is independent of  $\gamma$ .

Unfortunately, an exact solution for the frequency where  $|\bar{X}_n/X_n|$  is 0.707 would require the symbolic solution of a cubic equation. However, setting  $|\bar{X}_n/X_n| = 1$  results in an algebraically tractable expression. The exact solution is given in Equation (51) and a good approximation is given in Equation (52):

$$\omega T = \arccos(1 - \beta) \quad (51)$$

or

$$\omega T \approx \sqrt{2\beta} \quad (52)$$

Now observe that the frequency response from the point where  $|\bar{X}_n/X_n| = 1$  to the minimum is nearly a straight line on the log curve. Thus, approximating the response by a straight line should give a reasonable result. This procedure results in

$$\omega T \approx \sqrt{2\beta} 10^{-.15 \frac{\ln(\pi/\sqrt{2\beta})}{\ln(\alpha/2)}} \quad (53)$$

The same procedure was used to estimate the -6 dB point (-.15 in the exponent of (53) becomes -.3) with the results shown in Table 3.

TABLE 3. APPROXIMATE VERSUS EXACT  $\omega T$  FOR -6 DB

$\alpha$	.1	.3	.488
Approximate	.194	.65	1.18
Exact	.196	.65	1.20

An exact solution for  $\omega T$  which yields a maximum is algebraically intractable. Examination of Figure 1 reveals that the maximum of  $|\bar{X}_n/X_n|$  occurs near the midpoint of the 0 to  $|\bar{X}_n/X_n| = 1$  line segment. Thus,  $\omega T_{\max}$  is approximately 1/2 of the  $\omega T$  which yields  $|\bar{X}_n/X_n| = 1$ .

$$\omega T_{\max} = \sqrt{\beta/2} \quad (54)$$

Instead of calculating the maximum of  $|\bar{X}_n/X_n|$ , it is more convenient and useful to find the overshoot. The overshoot is defined overshoot = maximum - 1. At  $\omega T_{\max}$ , the overshoot is approximately

$$\text{overshoot} = \frac{M}{D} + \frac{M^2 + N^2}{2D^2} \quad (55)$$

where

$$M = \frac{3}{8} \beta^2 (1 - \alpha)(n + \beta/3) \quad (56)$$

$$N^2 = \frac{\beta^3 (1 - \alpha)^2 n^2}{16} \quad (57)$$

$$D = n^2 + \frac{\beta^3}{8} \quad (58)$$

with

$$n = \frac{\alpha\beta}{2} - \gamma + \frac{3}{8} \beta^2 \quad (59)$$

Note that increasing  $\gamma$  decreases  $n$  and thus increases the overshoot--provided that  $\gamma < \alpha\beta/2$ . Table 4 compares the exact overshoot with the approximation of Equation (55).

TABLE 4. APPROXIMATE AND EXACT OVERSHOOT

$\alpha$	.1	.3	.488
Exact Overshoot	0.269	0.232	0.192
Approximate Overshoot	0.275	0.261	0.237

The approximate and exact  $\omega_{T_{\max}}$  are compared in Table 5.

TABLE 5. APPROXIMATE AND EXACT  $\omega_{T_{\max}}$

$\alpha$	.1	.3	.488
Exact $\omega_{T_{\max}}$	0.0402	0.13325	0.2440
Approximate $\omega_{T_{\max}}$	0.0419	0.13338	0.2366

#### SUMMARY AND CONCLUSIONS

The frequency domain response of an  $\alpha$ - $\beta$ - $\gamma$  tracker provides information which is useful for the design of an  $\alpha$ - $\beta$ - $\gamma$  tracker. The Bode plot is very helpful if the  $\alpha$ - $\beta$ - $\gamma$  tracker is to be incorporated into a control system.

Equations (50), (53), and (55) present results which also are useful in the design of  $\alpha$ - $\beta$ - $\gamma$  trackers.

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